A Closed Form Solution to the Probability Hypothesis Density Smoother

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Abstract – *A closed form Gaussian mixture solution to the forward-backward Probability Hypothesis Density smoothing recursion is proposed. The key to the closed form solutions is the use of an alternative form of the backward propagation, together with terse yet suggestive notations that have natural interpretation in terms of measurement predictions. The closed form backward propagation together with the Gaussian mixture PHD filter as the forward pass form the Gaussian mixture PHD smoother. Closed form solutions to smoothing for single target are also derived.*

Keywords: PHD, Filtering, Smoothing, tracking, random sets, point processes, finite set statistics.

1 Introduction

Analytic *filtering* solutions such as the Kalman filter and Gaussian sum filter, for linear Gaussian and Gaussian mixture models, have opened up numerous research avenues and pervaded many application areas [9], [16], [1]. For general non-linear models, Sequential Monte Carlo (SMC) or particle filters have recently emerged as powerfull numerical approximations [7], [11], [4], [5].

Research in *smoothing* has experienced similar developments as in filtering, except for the smoothing analogue of the Gaussian sum filter. For general non-linear models, SMC approximations have been proposed for various smoothing schemes including *smoothing-while-filtering* [11], *forward-backward smoothing* [8], [4], (generalized) *two-filter smoothing* [3], and *block-based smoothing* [6]. For the special case of linear Gaussian model, an analytic smoothing solution exist in the form of the Kalman smoother [1]. However, for linear Gaussian mixture model, the Gaussian sum smoother–the smoothing analogue of the Gaussian sum filter–still remains elusive (see, for example [10]).

Filtering and smoothing are far more challenging in the multi-target realm since the number of targets varies randomly in time, obscured by clutter, detection uncertainty and data association uncertainty. The computational intractability is also much more severe in multi-target smoothing than filtering. The PHD filter [12], [13] is a multi-target filter that operates on the single-target state space and, consequently, avoids the high dimensionality that results from having multiple targets. Recently, a *forward-backward PHD smoother* has been proposed together with a sequential Monte Carlo implementation [15]. A rigorous derivation of the PHD smoother, using Finite Set Statistics (FISST) [12], [13] and Campbell's theorem, is given in [14]. Moreover, the backward PHD recursion in [14] is exact and does not require the smoothed PHD at the previous iteration to be Poisson as in [15]. The PHD is inherently multi-modal and, like the Gaussian sum smoother, a Gaussian mixture PHD smoother is not yet available.

In this paper, we propose a closed form solution to the PHD smoother under linear Gaussian assumption. Specifically, we derive analytic solutions for a generic backward recursion that covers backward smoothing for the PHD as well as for single-target in clutter, and subsequently, the Gaussian sum smoother. Two key innovations enabled the closed form smoothing solution. The first consists of an alternative form of the backward smoothing recursion, expressed in terms of a backward corrector that resembles the two-filter smoother [2]. The second is a set of terse yet suggestive notations that have natural interpretations in terms of measurement predictions. The proposed Gaussian mixture solution to the backward PHD recursion together with the GM-PHD filter [18], respectively, constitute the backward and forward passes of the Gaussian mixture PHD smoother.

2 Generic Gaussian mixture backward propagation

This section presents analytic solutions to a generic backward recursion which can be applied to PHD smoothing and various other backward smoothing recursions for single target.

2.1 Notation

The notation described in this subsection facilitates the derivation of the main results. We denote a Gaussian density with mean m and covariance P by $\mathcal{N}(\cdot; m, P)$, and for appropriate matrices H, R

$$\mathcal{N}_{H,R}(z;\zeta) = \mathcal{N}(z; H\zeta, R)$$

with the convention that $\mathcal{N}_{[\,],[\,]}(z;\zeta) = 1$. Consider a linear Gaussian dynamic and measurement model:

$$f_{k|k-1}(\zeta|x) = \mathcal{N}_{F_{k|k-1},Q_k}(\zeta; x) \tag{1}$$
$$g_k(z|\zeta) = \mathcal{N}_{H_k,R_k}(z;\zeta) \tag{2}$$

where $F_{k|k-1}$ is the state transition matrix, Q_k is the process noise covariance, H_k is the observation matrix, and R_k is the observation noise covariance. For $i, j > 0$ such that $i \geq j$, define

$$H_{i|i} = H_i, \tag{3}$$
$$R_{i|i} = R_i, \tag{4}$$
$$H_{i|j-1} = H_{i|j}F_{j|j-1}, \tag{5}$$
$$R_{i|j-1} = R_{i|j} + H_{i|j}Q_j H_{i|j}^T. \tag{6}$$

We also use the obvious short hand $[H,R]_{i|j}$ when we refer to the pair $H_{i|j}, R_{i|j}$ collectively. Thus, to construct $[H,R]_{i|j}$, we start with $[H,R]_{i|i}$ from (3)-(4), then repeatedly applying (5), (6) to construct $[H,R]_{i|i-1}$, and $[H,R]_{i|i-2}$, and so forth, until we reach $[H,R]_{i|j}$. Note that $H_{i|j} = H_i F_{i|i-1} \cdots F_{j+1|j}$.

For $j > 0$ define

$$H_{\emptyset|j} = [\,] \tag{7}$$
$$R_{\emptyset|j} = [\,]. \tag{8}$$

where $[\,]$ is the MATLAB notation for the null matrix, which satisfies

$$\begin{bmatrix} [\,] \\ M \end{bmatrix} = M$$

Consider the set of integers $I = \{i(1), ..., i(|I|)\}$, where $|I|$ denotes the cardinality of I, and by convention $i(1) > i(2) > ... > i(|I|)$. Given I and $j > 0$ with $i(|I|) > j$ or $I = \emptyset$, define

$$H_{I\cup\{j\}|j} = \begin{bmatrix} H_{I|j} \\ H_j \end{bmatrix}, \tag{9}$$
$$R_{I\cup\{j\}|j} = \begin{bmatrix} R_{I|j} & 0 \\ 0 & R_j \end{bmatrix}, \tag{10}$$

and similarly to (5), (6)

$$H_{I|j-1} = H_{I|j}F_{j|j-1} \tag{11}$$
$$R_{I|j-1} = R_{I|j} + H_{I|j}Q_j H_{I|j}^T \tag{12}$$

Note that $H_{\{j\}|j} = H_j$, $R_{\{j\}|j} = R_j$, $H_{\{i\}|j} = H_{i|j}$, and $R_{\{i\}|j} = R_{i|j}$. Again, we use the obvious short hand notation $[H,R]_{I|j}$ when we refer to the pair $H_{I|j}, R_{I|j}$ collectively.

For example, to construct $[H,R]_{I|j}$ we start with $[H,R]_{\{i(1)\}|i(2)}$ from (7), (8) and construct $[H,R]_{\{i(1),i(2)\}|i(2)}$ using(9), (10). The next step is to apply (11), (12) to construct $[H,R]_{\{i(1),i(2)\}|i(2)-1}$, and if necessary apply (11), (12) an appropriate number of times to reach $[H,R]_{\{i(1),i(2)\}|i(3)}$. This procedure is then repeated until we reach $[H,R]_{I|j}$.

For simplicity the following inner product notation is adopted throughout the paper

$$\langle f, g\rangle = \int f(x)g(x)dx$$

We also denote

$$\begin{aligned} z_I &= [z_{i(1)}^T, ..., z_{i(|I|)}^T]^T, \\ Z_I &= Z_{i(1)} \times ... \times Z_{i(|I|)}, \\ \sum_{z_I \in Z_I} f(z_I) &= \sum_{z_{i(1)} \in Z_{i(1)}} \cdots \sum_{z_{i(n)} \in Z_{i(n)}} f(z_{i(1)}, ..., z_{i(n)}), \end{aligned}$$

with the convention $\sum_{z_\emptyset \in Z_\emptyset} f(z_\emptyset) = 1$, this is not in conflict with the convention $\sum_{z\in\emptyset} f(z) = 0$.

2.2 Measurement Prediction Interpretation

The matrices defined by (5), (6) have a measurement prediction interpretation as follows. At time $j-1$, the statistics of the measurement z_{j-1} is captured by the measurement matrix H_{j-1} and measurement covariance R_{j-1}. The matrices $H_{i|j-1}$ and $R_{i|j-1}$ play analogous roles in the distribution of the measurement at time i, given the state at time $j-1$. The distribution $g_{j|j-1}(z_j|x_{j-1})$ of the measurement z_j at time j given x_{j-1} at time $j-1$ is (from the hidden Markov assumption, and Lemma 10 from Appendix A)

$$\begin{aligned} g_{j|j-1}(z_j|x_{j-1}) &= \int g_{j|j-1}(z_j|\zeta) f_{j|j-1}(\zeta|x_{j-1})d\zeta \\ &= \langle \mathcal{N}_{H_j,R_j}(z_j;\cdot), \mathcal{N}_{F_{j|j-1},Q_j}(\cdot; x_{j-1})\rangle \\ &= \mathcal{N}_{H_jF_{j|j-1},R_j+H_jQ_jH_j^T}(z_j; x_{j-1}) \\ &= \mathcal{N}_{H_{j|j-1},R_{j|j-1}}(z_j; x_{j-1}) \end{aligned}$$

Henceforth, $H_{j|j-1}$ and $R_{j|j-1}$ can be interpreted as the predicted measurement matrix and predicted measurement covariance to time j given the state at time $j-1$. Using the same arguments inductively, the matrices $H_{i|j-1}$ and $R_{i|j-1}$ defined by (5), (6) can be interpreted as the predicted measurement matrix and predicted measurement covariance to time i given the state at time $j-1$, i.e.

$$g_{i|j-1}(z_j|x_{j-1}) = \mathcal{N}_{H_{i|j-1},R_{i|j-1}}(z_i; x_{j-1})$$

In the same way, the matrices defined by (9), (10) have a measurement prediction interpretation. Suppose that the distribution $g_{I|j}(z_I|x_j)$ of the joint measurement z_I given a state x_j at time j, with $i(|I|) > j$, is a Gaussian

$$g_{I|j}(z_I|x_j) = \mathcal{N}_{H_{I|j},R_{I|j}}(z_I; x_j)$$

where $H_{I|j}$, $R_{I|j}$ are the predicted joint measurement matrix and joint measurement covariance from time j to I (true for singleton I). Since $i(|I|) > j$,

$$\begin{aligned}
g_{I\cup\{j\}|j}(z_{I\cup\{j\}}|x_j) &= \int p(z_{I\cup\{j\}}, x_I|x_j)dx_I \\
&= \int p(z_{I\cup\{j\}}|x_I, x_j)p(x_I|x_j)dx_I \\
&= g_j(z_j|x_j)\int g_I(z_I|x_I)p(x_I|x_j)dx_I \\
&= g_{I|j}(z_I|x_j)g_j(z_j|x_j) \\
&= \mathcal{N}_{H_{I|j},R_{I|j}}(z_I; x_j)\mathcal{N}_{H_j,R_j}(z_j; x_j)
\end{aligned}$$

Hence, the predicted joint measurements distribution $g_{I\cup\{j\}|j}(z_{I\cup\{j\}}|x_j)$ is Gaussian with mean and covariance given by (9), (10), i.e. the matrices (9), (10) can be interpreted as the prediction matrix and predicted measurement covariance matrix for the joint measurements in the index set $I \cup \{j\}$ from the state at time j. Moreover, from the hidden Markov assumptions, and Lemma 10 from Appendix A,

$$\begin{aligned}
g_{I|j-1}(z_I|x_{j-1}) &= \int g_{I|j}(z_I|\zeta)f_{j|j-1}(\zeta|x_{j-1})d\zeta \\
&= \langle \mathcal{N}_{H_{I|j},R_{I|j}}(z_I; \cdot), \mathcal{N}_{F_{j|j-1}Q_j}(\cdot; x_{j-1})\rangle \\
&= \mathcal{N}_{H_{I|j}F_{j|j-1},R_{I|j}+H_{I|j}Q_jH_{I|j}^T}(z_I; x_{j-1})
\end{aligned}$$

Hence, given a state x_{j-1} at time $j-1$, the joint measurement z_I is Gaussian distributed with joint predicted measurement matrix $H_{I|j-1}$ and joint measurement covariance $R_{I|j-1}$, defined by (11), (12).

2.3 Solutions to a Generic Backward Recursion

Consider a sequence of functions $B_{k|l}$, $k < l$, with $B_{l|l}(x) = 1$, generated according to a generic backward recursion of the following form

$$B_{k|l}(x) = q_{k+1} + p_{k+1}\langle B_{k+1|l}L_{k+1}(Z_{k+1}; \cdot), f_{k+1|k}(\cdot|x)\rangle \tag{13}$$

where

$$L_{k+1}(Z; x) = \alpha_{k+1} + \sum_{z\in Z} w_{k+1}(z)\mathcal{N}_{H_{k+1},R_{k+1}}(z; x) \tag{14}$$

$$f_{k+1|k}(\zeta|x) = \mathcal{N}_{F_{k+1|k},Q_{k+1}}(\zeta; x), \tag{15}$$

q_{k+1}, p_{k+1}, α_{k+1}, $w_{k+1}(z)$ are given non-negative constants and H_{k+1}, R_{k+1}, $F_{k+1|k}$, Q_{k+1} are (suitable) given matrices. This recursion covers a number of backward smoothing recursions including the PHD. A closed form solution to this generic backward recursion is given by the following result. The proof is given in Appendix B.

Proposition 1 *For $k < l$, if*

$$B_{k+1|l}(x) = \sum_{I\subseteq\{l:k+2\}} \sum_{z_I\in Z_I} w_{I|k+1}(z_I)\mathcal{N}_{[H,R]_{I|k+1}}(z_I; x), \tag{16}$$

then

$$\begin{aligned}
B_{k|l}(x) = q_{k+1} &+ \sum_{I\subseteq\{l:k+2\}} \sum_{z_I\in Z_I} p_{k+1}\alpha_{k+1}w_{I|k+1}(z_I)\mathcal{N}_{[H,R]_{I|k}}(z_I; x) \\
&+ \sum_{I\subseteq\{l:k+2\}} \sum_{z_I\in Z_I} \sum_{z_{k+1}\in Z_{k+1}} p_{k+1}w_{k+1}(z_{k+1})w_{I|k+1}(z_I) \\
&\times \mathcal{N}_{[H,R]_{I\cup\{k+1\}|k}}(z_{I\cup\{k+1\}}; x)
\end{aligned} \tag{17}$$

where the notation $\{l : k\}$ denotes $\{l, l-1, ..., k\}$.

The above result presupposes $B_{k+1|l}$ of the form (16). This is indeed the case. The following result provides a complete closed form solution for $B_{k|l}$. The proof is given in Appendix B.

Proposition 2 *For $k < l$,*

$$B_{k|l}(x) = \sum_{I\subseteq\{l:k+1\}} \sum_{z_I\in Z_I} w_{I|k}(z_I)\mathcal{N}_{[H,R]_{I|k}}(z_I; x) \tag{18}$$

$$\begin{aligned}
w_{I|k}(z_I) = &\left(\prod_{i=I_{|1|}+1}^{l} p_i\alpha_i + \sum_{j=I_{|1|}+1}^{l} q_j \prod_{i=I_{|1|}+1}^{j-1} p_i\alpha_i\right) \\
&\times \left(\prod_{i\in I} p_iw_i(z_i)\right)\left(\prod_{i\in\{I_{|1|}:k+1\}-I} p_i\alpha_i\right)
\end{aligned} \tag{19}$$

3 The Gaussian Mixture PHD smoother

This section details a closed form solution to the PHD smoother under linear Gaussian multi-target assumptions, using the generic backward recursion presented in the previous section. We also apply this the to derive a closed form smoothing solution to single target in clutter.

3.1 The PHD smoother

The forward-backward PHD smoother consists of a forward pass followed by a backward pass. The forward pass is, in fact, the PHD filter [12], and the backward has recently been proposed by [15], [14]. Denote

$$\begin{aligned}
v_{k|k} &= \text{filtered (updated) PHD at time } k \\
v_{k+1|k} &= \text{predicted PHD from time } k \text{ to } k+1 \\
v_{k|l} &= \text{smoothed PHD from time } l \text{ to } k\ (\ k < l)
\end{aligned}$$

$$\gamma_{k+1|k} = \text{PHD of birth at time } k+1$$
$$f_{k+1|k} = \text{single-target transition from time } k \text{ to } k+1$$
$$p_{S,k+1|k} = \text{probability of survival from } k \text{ to } k+1$$
$$q_{S,k+1|k} = 1 - p_{S,k+1|k}$$
$$g_{k+1} = \text{single-target likelihood at time } k+1$$
$$p_{D,k+1} = \text{probability of detection at } k+1$$
$$q_{D,k+1|k} = 1 - p_{D,k+1|k}$$
$$\kappa_{k+1} = \text{clutter PHD at } k+1$$

then, the forward-backward PHD smoother can be described by the following steps:

- PHD prediction [12], [13] (without spawning)

$$v_{k+1|k}(\zeta) = \gamma_{k+1|k}(\zeta) + \left\langle v_{k|k} p_{S,k+1|k}, f_{k+1|k}(\zeta|\cdot)\right\rangle \tag{20}$$

- PHD update [12], [13]

$$v_{k+1|k+1}(x) = v_{k+1|k}(x) L_{k+1}(Z_{k+1}; x) \tag{21}$$

where

$$L_{k+1}(Z;x) = \sum_{z\in Z} \frac{p_{D,k+1}(x) g_{k+1}(z|x)}{\kappa_{k+1}(z) + \left\langle p_{D,k+1} g_{k+1}(z|\cdot), v_{k+1|k}\right\rangle} + q_{D,k+1}(x) \tag{22}$$

- PHD backward smoothing [15], [14]:

$$v_{k|l}(x) = v_{k|k}(x) B_{k|l}(x) \tag{23}$$

where

$$B_{k|l}(x) = p_{S,k+1|k}(x) \left\langle \frac{v_{k+1|l}}{v_{k+1|k}}, f_{k+1|k}(\cdot|x)\right\rangle + q_{S,k+1|k}(x) \tag{24}$$

A particle implementation of the PHD smoother has been proposed in [15] using the particle PHD filter [17] for the forward pass, and a novel particle-based backward pass.

3.2 Linear Gaussian multi-target model

In addition to linear Gaussian dynamic and measurement for individual targets given by (1), (2), the linear Gaussian multi-target model assumes that:

- the survival and detection probabilities are state independent, i.e.

$$p_{S,k+1|k}(x) = p_{S,k+1|k}, \tag{25}$$
$$p_{D,k+1}(x) = p_{D,k+1}. \tag{26}$$

- the intensities of the birth RFS is a Gaussian mixture of the form

$$\gamma_{k+1|k}(x) = \sum_{i=1}^{J_{\gamma,k+1|k}} w_{\gamma,k+1|k}^{(i)} \mathcal{N}(x; m_{\gamma,k+1|k}^{(i)}, P_{\gamma,k+1|k}^{(i)}), \tag{27}$$

where $J_{\gamma,k+1|k}$, $w_{\gamma,k+1|k}^{(i)}$, $m_{\gamma,k+1|k}^{(i)}$, $P_{\gamma,k+1|k}^{(i)}$, $i = 1,\ldots,J_{\gamma,k+1|k}$, are given model parameters that determine the shape of the birth PHD.

Under linear Gaussian multi-target assumptions, it has been shown in [18] that if the initial PHD is a Gaussian mixture, then all subsequent predicted PHD and filtered PHD are also Gaussian mixtures. Moreover, a Gaussian mixture PHD filter was derived for propagating the predicted and filtered PHD [18]. We denote the Gaussian mixture predicted and filtered PHDs as follow

$$v_{k+1|k}(x) = \sum_{i=1}^{J_{k+1|k}} w_{k+1|k}^{(i)} \mathcal{N}(x; m_{k+1|k}^{(i)}, P_{k+1|k}^{(i)}), \tag{28}$$

$$v_{k|k}(x) = \sum_{i=1}^{J_{k|k}} w_{k|k}^{(i)} \mathcal{N}(x; m_{k|k}^{(i)}, P_{k|k}^{(i)}), \tag{29}$$

3.3 Gaussian mixture backward PHD recursion

The key to a closed form solution to the backward PHD recursion is the following recursion for the backward corrector term $B_{k|l}$

Proposition 3 *: Let $B_{l|l}(x) = 1$. Then for $k < l$,*

$$B_{k|l}(x) = p_{S,k+1|k}(x) \left\langle B_{k+1|l} L_{k+1}(Z_{k+1};\cdot), f_{k+1|k}(\cdot|x)\right\rangle + q_{S,k+1|k}(x) \tag{30}$$

Proof: It follows from (23) and (21) that

$$\frac{v_{k+1|l}}{v_{k+1|k}} = \frac{v_{k+1|k+1} B_{k+1|l}}{v_{k+1|k}} = L_{k+1}(Z_{k+1};\cdot) B_{k+1|l},$$

which upon substitution into (24) gives (30).

The forward PHD recursion (20), (21) and backward corrector recursion (30) can be thought of as a "2-filter" PHD smoother.

Under linear Gaussian multi-target assumptions, the pseudo likelihood $L_{k+1}(Z;\cdot)$ in the PHD smoother becomes

$$L_{k+1}(Z;\cdot) = \sum_{z\in Z} \frac{p_{D,k+1} \mathcal{N}_{H_{k+1},R_{k+1}}(z;\cdot)}{\kappa_{k+1}(z) + p_{D,k+1} r_{k+1}(z)} + q_{D,k+1} \tag{31}$$

where

$$r_{k+1}(z) = \left\langle \mathcal{N}_{H_{k+1},R_{k+1}}(z;\cdot), v_{k+1|k}\right\rangle = \sum_{j=1}^{J_{k+1|k}} w_{k+1|k}^{(j)} \mathcal{N}_{H_{k+1}, R_{k+1}+H_{k+1}P_{k+1|k}^{(j)}H_{k+1}^T}(z; m_{k+1|k}^{(j)})$$

This follows from (28), and (45) from Appendix A Thus, the backward correct recursion take on the generic form (13) and hence the closed form solutions to the backward corrector recursion follow from Propositions 1 and 2 with $q_i = q_{S,i|i-1}$, $p_i = p_{S,i|i-1}$,

$$\alpha_i = q_{D,i}, \;\; w_i(z_i) = \frac{p_{D,i}}{\kappa_i(z_i) + p_{D,i} r_i(z_i)}.$$

These solutions are summarized in the following propositions.

Proposition 4 *Under linear Gaussian multi-target assumption, if*

$$B_{k+1|l}(x) = q_{S,k+2|k+1} + \sum_{I\subseteq\{l:k+2\}}\sum_{z_I\in Z_I} w_{I|k+1}(z_I)\mathcal{N}_{[H,R]_{I|k+1}}(z_I;x)$$

then

$$\begin{aligned}B_{k|l}(x) = & q_{S,k+1|k} \\ & + \sum_{I\subseteq\{l:k+2\}}\sum_{z_I\in Z_I} p_{S,k+1|k}q_{D,k+1}w_{I|k+1}(z_I)\mathcal{N}_{[H,R]_{I|k}}(z_I;x) \\ & + \sum_{I\subseteq\{l:k+2\}}\sum_{z_I\in Z_I}\sum_{z_{k+1}\in Z_{k+1}} \frac{p_{S,k+1|k}p_{D,k+1}w_{I|k+1}(z_I)}{\kappa_{k+1}(z_{k+1})+p_{D,k+1}r_{k+1}(z_{k+1})} \\ & \times \mathcal{N}_{[H,R]_{I\cup\{k+1\}|k}}(z_{I\cup\{k+1\}};x)\end{aligned}$$

Proposition 5 *Under linear Gaussian multi-target assumption, the backward corrector $B_{k|l}$ for $k<l$ is given by*

$$B_{k|l}(x) = \sum_{I\subseteq\{l:k+1\}}\sum_{z_I\in Z_I} w_{I|k}(z_I)\mathcal{N}_{[H,R]_{I|k}}(z_I;x)$$

$$\begin{aligned}w_{I|k}(z_I) = & \left(\prod_{i=l}^{I_{|1|}+1} p_{S,i|i-1}q_{D,i} + \sum_{j=l}^{I_{|1|}+1} q_{S,j|j-1}\prod_{i=j-1}^{I_{|1|}+1} p_{S,i|i-1}q_{D,i}\right) \\ & \times\left(\prod_{i\in I}\frac{p_{S,i|i-1}p_{D,i}}{\kappa_i(z_i)+p_{D,i}r_i(z_i)}\right)\left(\prod_{i\in\{I_{|1|}:k+1\}-I} p_{S,i|i-1}q_{D,i}\right)\end{aligned}$$

Hence, using (23), (29), proposition 5, and then (46) from Appendix A, the closed form solution to the smoothed PHD is

Proposition 6 *Under linear Gaussian multi-target assumption, the smoothed PHD $v_{k|l}$ is a Gaussian mixture given by*

$$\begin{aligned}v_{k|l}(x) = & \sum_{I\subseteq\{l:k+1\}}\sum_{z_I\in Z_I}\sum_{i=1}^{J_{k|k}} w_{k|k}^{(i)}w_{I|k}(z_I)q_{k|k}^{(i)}(z_I) \\ & \times\mathcal{N}(x;\tilde{m}_{k|k}^{(i)}(z_I),\tilde{P}_{k|k}^{(i)})\end{aligned} \tag{32}$$

where

$$q_{k|k}^{(i)}(z_I) = \mathcal{N}_{H_{I|k},R_{I|k}+H_{I|k}P_{k|k}^{(i)}H_{I|k}^T}(z_I;m_{k|k}^{(i)}) \tag{33}$$

$$\tilde{m}_{k|k}^{(i)}(z_I) = m_{k|k}^{(i)} + K_{k|k}^{(i)}(z_I - H_{I|k}m_{k|k}^{(i)}) \tag{34}$$

$$\tilde{P}_{k|k}^{(i)} = (I - K_{k|k}^{(i)}H_{I|k})P_{k|k}^{(i)} \tag{35}$$

$$K_{k|k}^{(i)} = P_{k|k}^{(i)}H_{I|k}^T(H_{I|k}P_{k|k}^{(i)}H_{I|k}^T + R_{I|k})^{-1} \tag{36}$$

3.4 Closed form solutions to single-target smoothing

This subsection illustrates the utility of Propositions 1 and 2 through the derivation of a closed form solution for single target smoothing in the presence of detection uncertainty and clutter.

For each integer $k>0$, let

$$\begin{aligned}p_{k|k} &= \text{filtering density at time } k \\ p_{k+1|k} &= \text{prediction density from time } k \text{ to } k+1 \\ p_{k|l} &= \text{smoothing density from time } k \text{ to } l\ (\ k<l)\end{aligned}$$

Then under linear Gaussian assumptions–linear Gaussian dynamic and measurement, constant probability of detection, and Poisson clutter–if the initial prior is a Gaussian mixture, then all subsequent predicted and filtered densities are also Gaussian mixtures.

$$p_{k+1|k}(x) = \sum_{i=1}^{J_{k+1|k}} w_{k+1|k}^{(i)}\mathcal{N}(x;m_{k+1|k}^{(i)},P_{k+1|k}^{(i)}), \tag{37}$$

$$p_{k|k}(x) = \sum_{i=1}^{J_{k|k}} w_{k|k}^{(i)}\mathcal{N}(x;m_{k|k}^{(i)},P_{k|k}^{(i)}), \tag{38}$$

See [19] for the Gaussian mixture prediction and update. The single target forward-backward smoothing recursion consists of the following steps:

- prediction

$$p_{k+1|k}(\zeta) = \left\langle p_{k|k},\mathcal{N}_{F_{k+1|k},Q_{k+1}}(\zeta|\cdot)\right\rangle \tag{39}$$

- update

$$p_{k+1|k+1}(x) = p_{k+1|k}(x)L_{k+1}(Z_{k+1};x) \tag{40}$$

where

$$L_{k+1}(Z;x) = \frac{q_{D,k+1}\kappa_{k+1}^Z + p_{D,k+1}\sum_{z\in Z}\kappa_{k+1}^{Z-\{z\}}\mathcal{N}_{[H,R]_{k+1}}(z;x)}{r_{k+1}(Z)}$$

$$r_{k+1}(Z) = q_{D,k+1}\kappa_{k+1}^Z + p_{D,k+1}\sum_{z\in Z}\kappa_{k+1}^{Z-\{z\}}r_{k+1}(z)$$

$$r_{k+1}(z) = \sum_{j=1}^{J_{k+1|k}} w_{k+1|k}^{(j)}\mathcal{N}_{H_{k+1},R_{k+1}+H_{k+1}P_{k+1|k}^{(j)}H_{k+1}^T}(z;m_{k+1|k}^{(j)})$$

$$h^Z = \prod_{z\in Z} h(z)$$

with the convention $\kappa_{k+1}^{\emptyset}=1$, (even if $\kappa_{k+1}=0$),

- backward smoothing

$$p_{k|l}(x) = p_{k|k}(x)B_{k|l}(x) \tag{41}$$

where

$$B_{k|l}(x) = \left\langle\frac{p_{k+1|l}}{p_{k+1|k}},f_{k+1|k}(\cdot|x)\right\rangle \tag{42}$$

Similar to the PHD filter, the key to the closed form backward recursion is the following recursion for the backward corrector.

Proposition 7 *: Let $B_{l|l}(x) = 1$. Then for $k < l$,*

$$B_{k|l}(x) = \left\langle B_{k+1|l} L_{k+1}(z_{k+1};\cdot), f_{k+1|k}(\cdot|x)\right\rangle \quad (43)$$

Proof: It follows from (41) and (40) that

$$\frac{p_{k+1|l}}{p_{k+1|k}} = \frac{p_{k+1|k+1} B_{k+1|l}}{p_{k+1|k}} = L_{k+1}(z_{k+1};\cdot)B_{k+1|l},$$

which upon substitution into (42) gives (43).

The backward corrector recursion takes on the generic form (13) and hence the closed form solutions to the backward corrector recursion follow from Propositions 1 and 2 with $q_i = 0$, $p_i = 1$,

$$\alpha_i = \frac{q_{D,i}\kappa_i^{Z_i}}{r_i(Z_i)},\ w_i(z_i) = \frac{p_{D,i}\kappa_i^{Z_i-\{z_i\}}}{r_i(Z_i)}.$$

Proposition 8 *For $k < l$, under linear Gaussian assumptions, if*

$$B_{k+1|l}(x) = \sum_{I\in\{l:k+2\}} \sum_{z_I\in Z_I} w_{I|k+1}(z_I)\mathcal{N}_{[H,R]_{I|k+1}}(z_I;x)$$

then

$$\begin{aligned}B_{k|l}(x) &= \frac{\kappa_{k+1}^{Z_{k+1}} q_{D,k+1}}{r_{k+1}(Z_{k+1})} \sum_{I\in\{l:k+2\}} \sum_{z_I\in Z_I} w_{I|k+1}(z_I) \\ &\quad \times \mathcal{N}_{[H,R]_{I|k}}(z_I;x) \\ &\quad + \frac{p_{D,k+1}}{r_{k+1}(Z_{k+1})} \sum_{I\in\{l:k+2\}} \sum_{z_I\in Z_I} \sum_{z_{k+1}\in Z_{k+1}} \kappa_{k+1}^{Z_{k+1}-\{z_{k+1}\}} \\ &\quad \times w_{I|k+1}(z_I)\mathcal{N}_{[H,R]_{I\cup\{k+1\}|k}}(z_{I\cup\{k+1\}};x)\end{aligned}$$

Proposition 9 *Under linear Gaussian assumptions, the backward corrector $B_{k|l}$ for $k < l$ is given by*

$$\begin{aligned}B_{k|l}(x) &= \left(\prod_{i=k+1}^{l} r_i(Z_i)\right)^{-1} \\ &\quad \times \sum_{I\subseteq\{l:k+1\}} \sum_{z_I\in Z_I} \left(\prod_{j\in\{l:k+1\}-I} q_{D,j}\kappa_j^{Z_j}\right) \\ &\quad \times \left(\prod_{i\in I} p_{D,i}\kappa_i^{Z_i-\{z_i\}}\right)\mathcal{N}_{[H,R]_{I|k}}(z_I;x)\end{aligned}$$

Remark: The normalizing constant $\prod_{i=k+1}^{l} r_i(Z_i)$ is included for completeness, in practice there is no need to calculate it at all. Instead we normalize the weights of the Gaussian components of $p_{k|k}(x)B_{k|l}(x)$. Thus, it is not necessary to compute the prediction and filtering densities for time $k+1$ to l. In fact we only need (the Gaussian components of) the filtered density at time k, and predicted density to $k+1$.

The Gaussian sum smoother is a special case of the above proposition with $p_{D,i} = 1, \kappa_i = 0, Z_i = \{z_i\}$, where the backward corrector is given by

$$B_{k|l}(x) = \frac{\mathcal{N}_{[H,R]_{l:k+1|k}}(z_{l:k+1};x)}{r_{l:k+1}(z_{l:k+1})} \quad (44)$$

4 Appendix A

Lemma 10 *: Given F, Q and H, R of appropriate dimensions, and that Q and R are positive definite*

$$\langle \mathcal{N}_{H,R}(z;\cdot), \mathcal{N}(\cdot;Fx,Q)\rangle = \mathcal{N}_{HF,R+HQH^T}(z;x) \quad (45)$$

Lemma 11 *: Given H, R, m, and P of appropriate dimensions, and that R and P are positive definite,*

$$\mathcal{N}_{H,R}(z;x)\mathcal{N}(x;m,P) = \mathcal{N}(x;\tilde{m},\tilde{P})\mathcal{N}_{H,R+HPH^T}(z;m) \quad (46)$$

where

$$\tilde{m} = \tilde{m}(z,m) = m + K(z - Hm) \quad (47)$$

$$\tilde{P} = (I - KH)P \quad (48)$$

$$K = PH^T(HPH^T + R)^{-1} \quad (49)$$

Lemma 12 *Given I and $k \ge 0$, such that $i_{|I|} > j$, under linear Gaussian dynamic model (1),*

$$\begin{aligned}&\mathcal{N}_{[H,R]_{I|k}}(z_I;x) \\ &\quad = \left\langle \mathcal{N}_{[H,R]_{I|k+1}}(z_I;\cdot), f_{k+1|k}(\cdot|x)\right\rangle \quad (50)\\ &\mathcal{N}_{[H,R]_{I\cup\{k+1\}|k}}(z_{I\cup\{k+1\}};x) \\ &\quad = \left\langle \mathcal{N}_{[H,R]_{I|k+1}}(z_I;\cdot)\mathcal{N}_{[H,R]_{k+1}}(z_{k+1};\cdot), f_{k+1|k}(\cdot|x)\right\rangle \quad (51)\end{aligned}$$

Proof: For (50), using Lemma 10 and definitions of $H_{I|k}, R_{I|k}$ in (11), (12), we have

$$\begin{aligned}&\left\langle \mathcal{N}_{[H,R]_{I|k+1}}(z_I;\cdot), f_{k+1|k}(\cdot|x)\right\rangle \\ &= \left\langle \mathcal{N}_{H_{I|k+1},R_{I|k+1}}(z_I;\cdot), \mathcal{N}_{F_{k+1|k},Q_{k+1}}(\cdot;x)\right\rangle \\ &= \mathcal{N}_{H_{I|k+1}F_{k+1|k},R_{I|k+1}+H_{I|k+1}Q_{k+1}H_{I|k+1}^T}(z_I;x) \\ &= \mathcal{N}_{H_{I|k},R_{I|k}}(z_I;x) \\ &= \mathcal{N}_{[H,R]_{I|k}}(z_I;x)\end{aligned}$$

For (51), note from the product of Gaussian densities in z_I and z_{k+1} and definitions (9), (10) that

$$\begin{aligned}&\mathcal{N}_{[H,R]_{I|k+1}}(z_I;\zeta)\mathcal{N}_{[H,R]_{k+1}}(z_{k+1};\zeta) \\ &= \mathcal{N}_{H_{I|k+1},R_{I|k+1}}(z_I;\zeta)\mathcal{N}_{H_{k+1},R_{k+1}}(z_{k+1};\zeta) \\ &= \mathcal{N}_{H_{I\cup\{k+1\}|k+1},R_{I\cup\{k+1\}|k+1}}(z_{I\cup\{k+1\}};\zeta) \\ &= \mathcal{N}_{[H,R]_{I\cup\{k+1\}|k+1}}(z_{I\cup\{k+1\}};\zeta)\end{aligned}$$

Hence, it follows from (50) that

$$\begin{aligned}&\left\langle \mathcal{N}_{[H,R]_{I|k+1}}(z_I;\cdot)\mathcal{N}_{[H,R]_{k+1}}(z_{k+1};\cdot), f_{k+1|k}(\cdot|x)\right\rangle \\ &= \left\langle \mathcal{N}_{[H,R]_{I\cup\{k+1\}|k+1}}(z_{I\cup\{k+1\}};\cdot), f_{k+1|k}(\cdot|x)\right\rangle \\ &= \mathcal{N}_{[H,R]_{I\cup\{k+1\}|k}}(z_{I\cup\{k+1\}};x).\end{aligned}$$

5 Appendix B

Proof of Proposition 1: Using the generic pseudo likelihood (14), (50) and (51) we have

$$
\begin{aligned}
&B_{k+1|l}L_{k+1}\left(Z_{k+1};\cdot\right)\\
&=\sum_{I\subseteq\{l:k+2\}}\sum_{z_I\in Z_I}\alpha_{k+1}w_{I|k+1}(z_I)\mathcal{N}_{[H,R]_{I|k+1}}(z_I;\cdot)\\
&+\sum_{I\subseteq\{l:k+2\}}\sum_{z_I\in Z_I}\sum_{z_{k+1}\in Z_{k+1}}w_{k+1}(z_{k+1})w_{I|k+1}(z_I)\\
&\times\mathcal{N}_{[H,R]_{I|k+1}}(z_I;\cdot)\mathcal{N}_{H_{k+1},R_{k+1}}(z_{k+1};\cdot)\\
&=\sum_{I\subseteq\{l:k+2\}}\sum_{z_I\in Z_I}\alpha_{k+1}w_{I|k+1}(z_I)\mathcal{N}_{[H,R]_{I|k+1}}(z_I;\cdot)\\
&+\sum_{I\subseteq\{l:k+2\}}\sum_{z_I\in Z_I}\sum_{z_{k+1}\in Z_{k+1}}w_{k+1}(z_{k+1})w_{I|k+1}(z_I)\\
&\times\mathcal{N}_{[H,R]_{I\cup\{k+1\}|k+1}}(z_{I\cup\{k+1\}};\cdot)
\end{aligned}
$$

Hence, from the generic backward recursion (13),

$$
\begin{aligned}
B_{k|l}(x)&=q_{k+1}+p_{k+1}\left\langle B_{k+1|l}L_{k+1}(Z_{k+1};\cdot),f_{k+1|k}(\cdot|x)\right\rangle\\
&=q_{k+1}+\sum_{I\subseteq\{l:k+2\}}\sum_{z_I\in Z_I}p_{k+1}\alpha_{k+1}w_{I|k+1}(z_I)\\
&\times\left\langle\mathcal{N}_{[H,R]_{I|k+1}}(z_I;\cdot),f_{k+1|k}(\cdot|x)\right\rangle\\
&+\sum_{I\subseteq\{l:k+2\}}\sum_{z_I\in Z_I}\sum_{z_{k+1}\in Z_{k+v1}}p_{k+1}w_{k+1}(z_{k+1})w_{I|k+1}(z_I)\\
&\times\left\langle\mathcal{N}_{[H,R]_{I\cup\{k+1\}|k+1}}(z_{I\cup\{k+1\}};\cdot),f_{k+1|k}(\cdot|x)\right\rangle\\
&=q_{k+1}\\
&+\sum_{I\subseteq\{l:k+2\}}\sum_{z_I\in Z_I}p_{k+1}\alpha_{k+1}w_{I|k+1}(z_I)\mathcal{N}_{[H,R]_{I|k}}(z_I;x)\\
&+\sum_{I\subseteq\{l:k+2\}}\sum_{z_I\in Z_I}\sum_{z_{k+1}\in Z_{k+v1}}p_{k+1}w_{k+1}(z_{k+1})w_{I|k+1}(z_I)\\
&\times\mathcal{N}_{[H,R]_{I\cup\{k+1\}|k}}(z_{I\cup\{k+1\}};x)
\end{aligned}
$$

Proof of Proposition 2: The result holds for the initial step $k=l-1$, since applying Proposition 1 with $B_{l|l}=1$, gives

$$B_{l-1|l}(x)=q_l+p_l\alpha_l+\sum_{z_l\in Z_l}p_lw_l(z_l)\mathcal{N}_{[H,R]_{l|l-1}}(z_l;x)\quad(52)$$

$$=\sum_{I\subseteq\{l\}}\sum_{z_I\in Z_I}w_{I|l-1}(z_I)\mathcal{N}_{[H,R]_{I|l-1}}(z_I;x)\quad(53)$$

where

$$w_{I|l-1}(z_I)=\begin{cases}p_lw_l(z_l), I=\{l\}\\(q_l+p_l\alpha_l), I=\emptyset\end{cases}\quad(54)$$

Suppose that the result holds for $k+1$, i.e. for $J\subseteq\{l:k+2\}$

$$
\begin{aligned}
w_{J|k+1}(z_J)=&\left(\prod_{i=l}^{J_{|1|}+1}p_i\alpha_i+\sum_{j=l}^{J_{|1|}+1}q_j\prod_{i=j-1}^{J_{|1|}+1}p_i\alpha_i\right)\\
&\times\left(\prod_{i\in I}p_iw_i(z_i)\right)\left(\prod_{i\in\{J_{|1|}:k+2\}-I}p_i\alpha_i\right)\quad(55)
\end{aligned}
$$

then from Proposition 1, $B_{k|l}$ is given by (17) which can be expressed in the form (18) by setting

$$
w_{I|k}(z_I)=\begin{cases}q_{k+1}+p_{k+1}\alpha_{k+1}w_{\emptyset|k+1}(z_\emptyset),\ I=\emptyset\\p_{k+1}\alpha_{k+1}w_{I|k+1}(z_I),\ I\subseteq\{l:k+2\},I\neq\emptyset\\p_{k+1}w_{k+1}(z_{k+1})w_{J|k+1}(z_J),\ I=J\cup\{k+1\},J\subseteq\{l:k+2\}\end{cases}\quad(56)
$$

for $I\subseteq\{l:k+1\}$. It remains to show that (56) is identical to (19). For the case $I=J=\emptyset$, $J_{|1|}$ is taken as $k+1$ since we are only considering $J\subseteq\{l:k+2\}$ in (55)

$$
\begin{aligned}
w_{\emptyset|k+1}(z_\emptyset)&=\left(\prod_{i=l}^{k+2}p_i\alpha_i+\sum_{j=l}^{k+2}q_j\prod_{i=j-1}^{k+2}p_i\alpha_i\right)\\
w_{I|k}(z_I)&=q_{k+1}+p_{k+1}\alpha_{k+1}\\
&\times\left(\prod_{i=l}^{k+2}p_i\alpha_i+\sum_{j=l}^{k+2}q_j\prod_{i=j-1}^{k+2}p_i\alpha_i\right)\\
&=q_{k+1}+\prod_{i=l}^{k+1}p_i\alpha_i+\sum_{j=l}^{k+2}q_j\prod_{i=j-1}^{k+1}p_i\alpha_i\\
&=\prod_{i=l}^{k+1}p_i\alpha_i+\sum_{j=l}^{k+1}q_j\prod_{i=j-1}^{k+1}p_i\alpha_i\quad(57)
\end{aligned}
$$

For the case $I\subseteq\{l:k+2\},I\neq\emptyset$

$$
\begin{aligned}
w_{I|k}(z_I)&=p_{k+1}\alpha_{k+1}\left(\prod_{i=l}^{I_{|1|}+1}p_i\alpha_i+\sum_{j=l}^{I_{|1|}+1}q_j\prod_{i=j-1}^{I_{|1|}+1}p_i\alpha_i\right)\\
&\times\left(\prod_{i\in I}p_iw_i(z_i)\right)\left(\prod_{i\in\{I_{|1|}:k+2\}-I}p_i\alpha_i\right)\\
&=\left(\prod_{i=l}^{I_{|1|}+1}p_i\alpha_i+\sum_{j=l}^{I_{|1|}+1}q_j\prod_{i=j-1}^{I_{|1|}+1}p_i\alpha_i\right)\\
&\times\left(\prod_{i\in I}p_iw_i(z_i)\right)\left(\prod_{i\in\{I_{|1|}:k+1\}-I}p_i\alpha_i\right)
\end{aligned}
$$

The third case $I=J\cup\{k+1\},J\subseteq\{l:k+2\}$ (Note that $J_{|1|}=I_{|1|}$ unless J is empty, in which case the result holds trivially)

$$
\begin{aligned}
w_{J|k+1}(z_J)&=\left(\prod_{i=l}^{I_{|1|}+1}p_i\alpha_i+\sum_{j=l}^{I_{|1|}+1}q_j\prod_{i=j-1}^{I_{|1|}+1}p_i\alpha_i\right)\\
&\times\left(\prod_{i\in J}p_iw_i(z_i)\right)\left(\prod_{i\in\{I_{|1|}:k+2\}-J}p_i\alpha_i\right)
\end{aligned}
$$

$$
\begin{aligned}
w_{I|k}(z_I) &= p_{k+1} w_{k+1}(z_{k+1}) \\
&\times \left(\prod_{i=l}^{I_{|1|}+1} p_i\alpha_i + \sum_{j=l}^{I_{|1|}+1} q_j \prod_{i=j-1}^{I_{|1|}+1} p_i\alpha_i \right) \\
&\times \left(\prod_{i\in J} p_i w_i(z_i) \right) \left(\prod_{i\in\{I_{|1|}:k+2\}-J} p_i\alpha_i \right) \\
&= \left(\prod_{i=l}^{I_{|1|}+1} p_i\alpha_i + \sum_{j=l}^{I_{|1|}+1} q_j \prod_{i=j-1}^{I_{|1|}+1} p_i\alpha_i \right) \\
&\times \left(\prod_{i\in J\cup\{k+1\}} p_i w_i(z_i) \right) \left(\prod_{i\in\{J_{|1|}:k+2\}-J} p_i\alpha_i \right) \\
&= \left(\prod_{i=l}^{I_{|1|}+1} p_i\alpha_i + \sum_{j=l}^{I_{|1|}+1} q_j \prod_{i=j-1}^{I_{|1|}+1} p_i\alpha_i \right) \\
&\times \left(\prod_{i\in J\cup\{k+1\}} p_i w_i(z_i) \right) \\
&\times \left(\prod_{i\in\{I_{|1|}:k+2\}\cup\{k+1\}-J\cup\{k+1\}} p_i\alpha_i \right) \\
&= \left(\prod_{i=l}^{I_{|1|}+1} p_i\alpha_i + \sum_{j=l}^{I_{|1|}+1} q_j \prod_{i=j-1}^{I_{|1|}+1} p_i\alpha_i \right) \\
&\times \left(\prod_{i\in I} p_i w_i(z_i) \right) \left(\prod_{i\in\{I_{|1|}:k+1\}-I} p_i\alpha_i \right)
\end{aligned}
$$

Hence the result follows by induction.

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Acknowledgement: This research was supported under Australian Research Councils Discovery Projects funding schemes and DP0878158 and DP0989007.